ON THE INDENTATION OF A RIGID WEDGE INTO A SEMIPLANE UNDER CONDITIONS OF STEADY CREEP

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The problem considered here is that of the indentation of a rigid wedge into a semiplane under conditions of steady creep, with a power law between the stresses and the strain velocities*. The problem is solved by a method developed in [1] and [2].

We have a plane wedge and take the origin of coordinates at the point O and direct the x and y axes as shown in Fig. 1. Then the equation of the wedge surface will be

$$f(x) = k|x| \qquad (k = \tan\varphi) \tag{1}$$

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* We note that the choice of the theory of steady creep with the power law (2) was not influenced by the existence of a method of solution. One might have started with the theory of strain-hardening, of which the equation is

$$\varepsilon_i = A\sigma_i^{\,p/(q+1)}t^{1/(q+1)} \qquad (a,\ p,\ q = \text{const})$$

or with the theory of plastic heredity, of which the equation is

$$\varphi^*(\varepsilon_i) = K\varepsilon_i^{\mu}(t) = \sigma_i(t) - \int_{\tau_1}^{t}\sigma_i(\tau)\frac{\partial c(t,\tau)}{\partial\tau}d\tau$$

Here $C(t, \tau)$ is a measure of the creep of the materia, $\phi^*(\epsilon_i)$ is a certain function characterizing the nonlinear relation between stress and strain. The contact problem is solved here under conditions of steady creep only for simplicity of the exposition.

Here ϕ is the angle between the wedge boundary and the x-axis.

We assume that a power law exists between the creep stress intensity σ_i and the strain velocity ϵ_i

$$\sigma_i = K\varepsilon_i^{\mu} \tag{2}$$

where K is a creep constant and μ is a creep exponent determined from tests on simple creep; $0 < \mu < 1$.

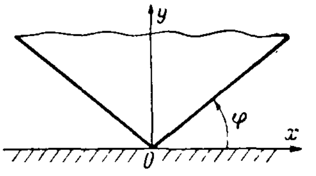

Fig. 1.

Let the initial contact of the wedge in the xy plane proceed from a point, which we take as origin. We suppose further that the region of contact after compression will be the interval $-a \leqslant x \leqslant a$ on the x-axis.

Then, as was shown in [1], the integral equation for the problem takes the form

$$\int_{-a}^{a} \frac{p(s)\,ds}{|s-x|^{1-\mu}} = [\gamma - \alpha|x|]^{\mu} \qquad (-a \leqslant x \leqslant a), \qquad \alpha = \frac{k}{A} \tag{3}$$

Here $p(s)$ is the intensity of pressure at the contact, γ a certain arbitrary constant to be determined later, A a known constant determined from [1]

$$A = \frac{(m-2)\sin(l\pi/2)}{K^m(m-1)J^m(\mu)} \qquad \left(l = \frac{\sqrt{2\mu-1}}{\mu}\right)$$
$$J(\mu) = 4l^{\mu}\int_0^{\pi/2}(\cos l\theta)^{\mu}\cos\theta d\theta \qquad \left(m = \frac{1}{\mu}\right) \qquad \text{for } \mu > \tfrac{1}{2} \tag{4}$$

and

$$A = \frac{(m-2)\sinh(\lambda\pi/2)}{K^m(m-1)J^m(\mu)} \qquad \left(\lambda = \frac{\sqrt{1-2\mu}}{\mu}\right)$$
$$J(\mu) = 4\lambda^{\mu}\int_0^{\pi/2}(\operatorname{ch}\lambda\theta)^{\mu}\cosh\theta d\theta \qquad \text{for } \mu < \tfrac{1}{2} \tag{5}$$

In accordance with [3] the general solution of the integral equation (3) will be

$$p(x) = \frac{1}{2M'(a)}\left[\frac{d}{da}\int_{-a}^{a} g(s,a)F(s,\gamma)\,ds\right]g(x,a) - \tag{6}$$

$$-\frac{1}{2}\int_x^a g(x,u)\frac{d}{du}\left[\frac{1}{M'(u)}\frac{d}{du}\int_{-u}^{u} g(s,u)F(s,\gamma)\,ds\right]du -$$

$$-\frac{1}{2}\frac{d}{dx}\int_x^a \frac{g(x,u)}{M'(u)}\left[\int_{-a}^{a} g(s,u)F'(s,\gamma)\,ds\right]du$$

Here

$$M(u) = \int_0^u g(s,u)\,ds \qquad (0 \leqslant u \leqslant a) \tag{7}$$

$$g(s,u) = \frac{\sin(\pi\mu/2)}{\pi\sqrt{(u^2-x^2)^\mu}}, \qquad F(s,\gamma) = [\gamma - \alpha|s|]^\mu \tag{8}$$

By virtue of the fact that $g(s, u)$ is an even function and that $F'(s, \gamma)$ is an odd function, as follows from Equation (8), the last term on the right side of Formula (6) will drop out, and the expression for $p(x)$ after certain transformations takes the form

$$p(x) = K(\mu)\left\{\frac{a^\mu \Phi_1'(a,\gamma)}{\sqrt{(a^2-x^2)^\mu}} - \int_x^a \frac{du}{\sqrt{(u^2-x^2)^\mu}}\frac{d}{du}[u^\mu \Phi_1'(u,\gamma)]\right\} \tag{9}$$

where

$$K(\mu) = \frac{1}{2\sqrt{\pi}\pi^2}\Gamma\left(\frac{1-\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)\sin^2\frac{\pi\mu}{2} \tag{10}$$

$$\Phi_1(u,\gamma) = \int_0^u \frac{F(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^\mu}}, \qquad \Phi_1'(u,\gamma) = \frac{d}{du}\int_0^u \frac{F(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^\mu}} \tag{11}$$

Upon making the substitution $s = u \sin\phi$, the expression for $\Phi_1(u, \gamma)$ may be represented as a definite integral

$$\Phi_1(u,\gamma) = u^{1-\mu}\int_0^{\pi/2} F(u\sin\varphi,\gamma)\cos^{1-\mu}\varphi\,d\varphi \tag{12}$$

By assuming a continuous and bounded derivative for the function $F(s, \gamma)$ when $s > 0$, we get after differentiating under the integral sign

$$u\Phi_1'(u,\gamma) = (1-\mu)\Phi_1(u,\gamma) + \int_0^u \frac{sF'(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^\mu}} \tag{13}$$

From Equation (12) it is not difficult to convince oneself at once that

$$\frac{d}{du}[u^\mu \Phi_1'(u,\gamma)] = u^{\mu-1}\frac{d}{du}\int_0^u \frac{sF'(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^\mu}} \tag{14}$$

After integration by parts on the right side of Equation (14) and differentiation of the resulting expression with respect to u we shall have

$$\frac{d}{du}\left[u\Phi_1'(u,\gamma)\right] = u^{\mu}\int_0^u \frac{F''(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^{\mu}}} + F'(0,\gamma) \tag{15}$$

Substitution of this expression in Equation (9) gives

$$p(x) = K(\mu)\left\{\frac{a^{\mu}\Phi_1'(a,\gamma)}{\sqrt{(a^2-x^2)^{\mu}}} - \int_a^x \frac{u^{\mu}\,du}{\sqrt{(u^2-x^2)^{\mu}}}\int_0^u \frac{F''(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^{\mu}}} - F'(0,\gamma)\int_x^a \frac{du}{\sqrt{(u^2-x^2)^{\mu}}}\right\} \tag{16}$$

For the derivative $F'(0, \gamma)$ one must evidently take the right limit

$$F'(0,\gamma) = F'(+0,\gamma) = -\alpha\mu\gamma^{\mu-1} \tag{17}$$

We have finally

$$p(x) = K(\mu)\left\{\frac{a^{\mu}\Phi_1'(a,\gamma)}{\sqrt{(a^2-x^2)^{\mu}}} + \alpha^2\mu(1-\mu)\int_a^x \frac{u^{\mu}\,du}{\sqrt{(u^2-x^2)^{\mu}}}\int_0^u \frac{(\gamma-\alpha s)^{\mu-2}\,ds}{\sqrt{(u^2-s^2)^{\mu}}} + \right.$$
$$\left. + \alpha\mu\gamma^{\mu-1}\int_x^a \frac{du}{\sqrt{(u^2-x^2)^{\mu}}}\right\} \tag{18}$$

1. We consider the case when the line of contact is known. The first term of Formula (18) represents the solution with singularities at $x = \pm a$ and is retained only if the width of contact is $2a$. The constant γ is determined from the equation of equilibrium

$$P = 2\int_0^a p(x)\,dx \tag{19}$$

Substituting Expression (18) for $p(x)$ into Equation (19), we obtain

$$P = 2K(\mu)\left\{a\Phi_1'(a,\gamma)\frac{\sin(\pi\mu/2)}{2(1-\mu)K(\mu)\pi} + \alpha\mu\gamma^{\mu-1}\int_0^a dx\int_x^a \frac{du}{\sqrt{(u^2-x^2)^{\mu}}} - \right.$$
$$\left. - \int_0^a dx\int_x^a \frac{u^{\mu}du}{\sqrt{(u^2-x^2)^{\mu}}}\int_0^u \frac{F''(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^{\mu}}}\right\} \tag{20}$$

Here the integral equality

$$J_1 = \int_0^u \frac{ds}{\sqrt{(u^2-s^2)^{\mu}}} = \frac{\sin(\pi\mu/2)}{2(1-\mu)K(\mu)\pi}\,u^{1-\mu} \tag{21}$$

proves to be useful.

By changing the order of integration in the second term of Equation (20), we find

$$\int_0^a dx \int_x^a \frac{du}{\sqrt{(u^2-x^2)^\mu}} = \int_0^a du \int_0^u \frac{dx}{\sqrt{(u^2-x^2)^\mu}} = \frac{\sin(\pi\mu/2)}{2(1-\mu)(2-\mu)K(\mu)\pi} a^{2-\mu} \quad (22)$$

Then Equation (20) takes the form

$$P = \frac{1}{(1-\mu)\pi} \sin\frac{\pi\mu}{2}\left[a\Phi_1'(a,\gamma) + \frac{\alpha\mu\gamma^{\mu-1}}{2-\mu}a^{2-\mu}\right] - $$
$$- 2K(\mu)\int_0^a dx \int_x^a \frac{u^\mu\,du}{\sqrt{(u^2-x^2)^\mu}} \int_0^u \frac{F''(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^\mu}} \quad (23)$$

By changing the order of integration in the last term of Expression (23) and making use of the integral relation (21), we find

$$J_2 = \int_0^a dx \int_x^a \frac{u^\mu\,du}{\sqrt{(u^2-x^2)^\mu}} \int_0^u \frac{F''(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^\mu}} = \frac{\sin(\pi\mu/2)}{2(1-\mu)K(\mu)\pi}\int_0^a u\,du \int_0^u \frac{F''(s,\gamma)\,ds}{\sqrt{(u^2-s^2)^\mu}} \quad (24)$$

Furthermore, by application of the Dirichlet formula and noting that

$$\int_s^a \frac{u\,du}{\sqrt{(u^2-s)^\mu}} = \frac{1}{2-\mu}(a^2-s^2)^{1-\frac{\mu}{2}} \quad (25)$$

the expression for J_2 may be given the form

$$J_2 = \frac{\sin(\pi\mu/2)}{2(1-\mu)(2-\mu)K(\mu)\pi}\int_0^a (a^2-s^2)^{1-\frac{\mu}{2}} F''(s,\gamma)\,ds. \quad (26)$$

After substitution of the value of J_2 from Equation (26) into Equation (23) we obtain

$$P = \frac{\sin(\pi\mu/2)}{(1-\mu)\pi}\left\{a\Phi_1'(a,\gamma) + \frac{\alpha\mu\gamma^{\mu-1}}{2-\mu}a^{2-\mu} - \frac{1}{2-\mu}\int_0^a (a^2-s^2)^{1-\frac{\mu}{2}}F''(s,\gamma)\,ds\right\} \quad (27)$$

If the last term in Equation (13) is integrated by parts, using expression (17), then it takes the form

$$u\Phi_1'(u,\gamma) = (1-\mu)\Phi_1(u,\gamma) - \frac{\alpha\mu\gamma^{\mu-1}}{2-\mu}u^{2-\mu} + \frac{1}{2-\mu}\int_0^u (u^2-s^2)^{1-\frac{\mu}{2}}F''(s,\gamma)\,ds \quad (28)$$

With this relation Equation (27) may now be presented in the form

$$\Phi_1(a,\gamma) = \frac{P\pi}{\sin(\pi\mu/2)} \quad (29)$$

where

$$\Phi_1(a,\gamma) = \int_0^a \frac{F(s,\gamma)\,ds}{\sqrt{(a^2-s^2)^\mu}}, \qquad F(s,\gamma) = (\gamma - \alpha|s|)^\mu \quad (30)$$

We obtain, upon substituting (30) into (29)

$$\int_0^a \frac{(\gamma - \alpha s)^{\mu}\, ds}{\sqrt{(a^2 - s^2)^{\mu}}} = \frac{P\pi}{\sin(\pi\mu/2)} \tag{31}$$

Thus, if the width of the contact is given as $2a$, the constant γ in Formula (18) is given by Equation (31). This equation may be solved by numerical methods.

2. We pass on to consideration of the second case, in which the length of contact is unknown. Evidently the value of γ in this case is also unknown. Two equations are required to determine the two unknowns a and γ. One equation is obtained from the requirement that the first term in Formula (18), representing the solution with singularities, vanishes; i.e.

$$\Phi_1'(a, \gamma) = \frac{d}{da}\int_0^a \frac{(\gamma - \alpha\,|\,s\,|)^{\mu}}{\sqrt{(u^2 - s^2)^{\mu}}}\, ds = 0 \tag{32}$$

and the other equation may be obtained with the aid of the equation of equilibrium (19). By substitution of the expression for $p(x)$ from (18) into (19), we obtain after certain transformations and taking account of (32),

$$\Phi_1(a, \gamma) = \int_0^a \frac{(\gamma - \alpha\,|\,s\,|)^{\mu}\, ds}{\sqrt{(a^2 - s^2)^{\mu}}} = \frac{P\pi}{\sin(\pi\mu/2)} \tag{33}$$

With the aid of relation (27), Equation (32) may be put in the form

$$\alpha\mu\gamma^{\mu-1} a^{2-\mu} + \alpha^2\mu(1-\mu)\int_0^a \frac{(a^2 - s^2)^{1-\frac{\mu}{2}}\, ds}{(\gamma - \lambda\,|\,s\,|)^{2-\mu}} = (1-\mu)(2-\mu)\frac{P\pi}{\sin(\pi\mu/2)} \tag{34}$$

The values of a and γ may be determined from the system of Equations (33) and (34). Substitution of the values so found into (18) gives the expression for $p(x)$.

In certain cases it is more convenient to present the expression for $p(x)$ in the form

$$p(x) = K(\mu)\left\{\frac{1}{\sqrt{(a^2 - x^2)^{\mu}}}\left[\frac{1-\mu}{a^{1-\mu}}\frac{P\pi}{\sin(\pi\mu/2)} - \right.\right.$$

$$\left. - \frac{a}{2-\mu}\alpha\mu\gamma^{\mu-1} - \frac{a^{\mu-1}}{2-\mu}\alpha^2\mu(1-\mu)\int_0^a (a^2 - s^2)^{1-\frac{\mu}{2}}\frac{ds}{(\gamma - \alpha s)^{2-\mu}}\right] +$$

$$\left. + \alpha^2\mu(1-\mu)\int_a^x \frac{u^{\mu}du}{\sqrt{(u^2 - x^2)^{\mu}}}\int_0^u \frac{(\gamma - \alpha s)^{\mu-2}}{\sqrt{(u^2 - s^2)^{\mu}}}\, ds + \alpha\mu\gamma^{\mu-1}\int_x^a \frac{du}{\sqrt{(u^2 - x^2)^{\mu}}}\right\} \tag{35}$$

Here the relations (28) and (29) have been used.

As an example we consider the contact problem of the indentation of a rigid wedge into a semiplane, the material of which exhibits the property of steady creep. We consider the case when the width of contact is known (Fig. 2).

We have the following initial data:

$$\varphi = \frac{\pi}{6}, \quad a = 2\ cm, \quad P = 100\ \text{kg/cm}$$

The relation between the stress intensity and the strain velocity may be presented in a form agreeing with experimental data [4]:

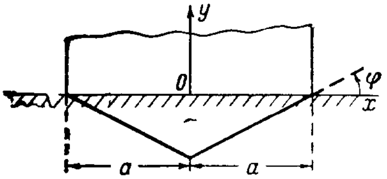

Fig. 2.

for chrome-nickel-tungsten steel EU123

$$\sigma_i = 0.1797 \cdot 10^9\, \varepsilon_i^{0.82} \quad (\mu = 0.82 > 0.5) \quad (36)$$

for carbon steel,

$$\sigma_i = 0.44065 \cdot 10^4\, \varepsilon_i^{0.33} \quad (\mu = 0.33 < 0.5) \quad (37)$$

We obtain the value of γ for these materials from the numerical solution of Equation (31), and with these values of γ we calculate the values of $p(x)$ by Formula (35) at the points $x = 0$ and $x = a/2$:

for chrome-nickel-tungsten steel EU123

$$(36)\ \gamma = -0.789105 \cdot 10^{10}, \quad p(0) = \infty, \quad p(a/2) = 0.196806 \cdot 10^{10}$$

for carbon steel,

$$(37)\ \gamma = 2.876589 . 10^{10}, \quad p(0) = \infty, \quad p(a/2) = 0.784078 \cdot 10^{7}$$

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